Limit distributions of expanding translates of certain orbits on homogeneous spaces

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Abstract. Let L be a Lie group and Λ a lattice in L. Suppose G is a non-compact simple Lie group realized as a Lie subgroup of L and $\overline{G\Lambda} = L$. Let $a \in G$ be such that $\mathrm{Ad}a$ is semisimple and not contained in a compact subgroup of $\mathrm{Aut}(\mathrm{Lie}(G))$. Consider the expanding horospherical subgroup of G associated to a defined as $U^+ = \{g \in G : a^{-n}ga^n \to e \text{ as } n \to \infty\}$. Let Ω be a nonempty open subset of U^+ and $n_i \to \infty$ be any sequence. It is showed that $\overline{\cup_{i=1}^{\infty} a^{n_i}\Omega\Lambda} = L$. A stronger measure theoretic formulation of this result is also obtained. Among other applications of the above result, we describe G-equivariant topological factors of $L/\Lambda \times G/P$, where the real rank of G is greater than 1, P is a parabolic subgroup of G and G acts diagonally. We also describe equivariant topological factors of unipotent flows on finite volume homogeneous spaces of Lie groups.

Keywords. Limit distributions; unipotent flow; horospherical patches; symmetric subgroups; equivariant topological factors

1 Introduction

Let G be a connected semisimple Lie group with no compact factors and of $\mathbb{R}$-rank ≥ 2, P a parabolic subgroup of G, and Γ an irreducible lattice in G. It was proved by Margulis [M1] that if $\phi : G/P \to Y$ is a measure class preserving Γ-equivariant factor of G/P then there exist a parabolic subgroup Q containing P and a measurable isomorphism $\psi : Y \to G/Q$ such that $\psi \circ \phi$ is the canonical quotient map. The topological analogue of this result was obtained by Dani [D3], who proved that, in the above notation, if ϕ is continuous then ψ can be chosen to be a homeomorphism. On the other hand in the result of Margulis was generalized by Zimmer [Z1] in the measure theoretic category. This result was later used in [SZ] for describing faithful and properly ergodic finite measure preserving G-actions. It was suggested by Stuck [St] that the following question, which is a topological analogue of Zimmer's result, is of importance for studying locally free minimal G-actions.

Question 1.1 *Let G be a simple Lie group of $\mathbb{R}$-rank≥ 2. Suppose that G acts minimally and locally freely on a compact Hausdorff space X. Suppose there are G-equivariant continuous surjective maps $X \times G/P \xrightarrow{\phi} Y \xrightarrow{\psi} X$ such that $\psi \circ \phi$ is the projection onto X, where G acts diagonally on $X \times G/P$. Does there exist a parabolic subgroup Q containing P and a G-equivariant homeomorphism $\rho : Y \to X \times G/Q$ such that $\rho \circ \phi$ is the canonical quotient map?*

The above mentioned result of Dani says that this question has the affirmative answer if $X = G/\Gamma$, Γ being a lattice in G. In this paper we consider the case when G is a Lie subgroup of a Lie group L acting on $X = L/\Lambda$ by translations, Λ being a lattice in L. To analyze this case we follow the method of the proof of Dani [D3]. To adapt Dani's proof for the general case one needs the following theorem 1.1, which is a nontrivial generalization of its particular case of $L = G$ (cf. [D3, Lemma 1.1]). Its proof involves, in an essential way, Ratner's theorem [Ra1] on classification of finite ergodic invariant measures of unipotent flows on homogeneous spaces.

For the results stated in the introduction, let L denote a connected Lie group, Λ a lattice in L, $\pi : L \to L/\Lambda$ the natural quotient map, and μ_L the (unique) L-invariant probability measure on L/Λ.

Theorem 1.1 *Let G be a connected semisimple Lie group. Let $a \in G$ be a semi-simple element; that is,* $\mathrm{Ad}(a)$ *is a semi-simple endomorphism of the Lie algebra of G. Consider the expanding horospherical subgroup U^+ of G associated to a which is defined as*

$$U^+ = \{u \in G : a^{-n}ua^n \to e \text{ as } n \to \infty\}.$$

Assume that U^+ is not contained in any proper closed normal subgroup of G.

Suppose that G is realized as a Lie subgroup of L and that $\overline{\pi(G)} = L/\Lambda$. Then

$$\overline{\pi\left(\cup_{n=1}^{\infty} a^n U^+\right)} = L/\Lambda.$$

In particular, if P is any parabolic subgroup of G and $\overline{\pi(G)} = L/\Lambda$, then $\overline{\pi(P)} = L/\Lambda$.

In the case of $L = G$ this result is well-known (see [DR, Prop. 1.5]). Actually theorem 1.1 is a straightforward consequence of a technically much stronger result stated later in the introduction as theorem 1.4.

Using the techniques of [D3] along with theorem 1.1 and the result of Ratner [Ra2] on closures of orbits of unipotent flows on finite volume homogeneous spaces, in the next result we provide an affirmative answer to Question 1.1 in case when $X = L/\Lambda$. In this case we are able relax certain other conditions in the question as well.

Theorem 1.2 *Let G be a semisimple Lie group of* $\mathbb{R}$-rank ≥ 2 *and with finite center. Suppose that G is realized as a Lie subgroup of L such that the G-action is ergodic with respect to μ_L, and that $\overline{G_1 x} = \overline{Gx}$ for any $x \in L/\Lambda$ and any closed normal connected subgroup G_1 of G such that* $\mathbb{R}$-rank$(G/G_1) \leq 1$. *Let P be a parabolic subgroup of G and consider the diagonal action of G on $L/\Lambda \times G/P$. Let Y be a Hausdorff space with a continuous G-action and $\phi : L/\Lambda \times G/P \to Y$ a continuous G-equivariant map. Then there exist a parabolic subgroup $Q \supset P$, a locally compact Hausdorff space X with a continuous G-action, a continuous surjective G-equivariant map $\phi_1 : L/\Lambda \to X$, and a continuous G-equivariant map $\psi : X \times G/Q \to Y$ such that the following holds:*

1. *If we define $\rho : L/\Lambda \times G/P \to X \times G/Q$ as $\rho(x, gP) = (\phi_1(x), gQ)$ for all $x \in L/\Lambda$ and $g \in G$, then*
$$\phi = \psi \circ \rho.$$
2. *There exists an open dense G-invariant set $X_0 \subset L/\Lambda$ such that if we put $Z_0 = \phi_1(X_0) \times G/Q$ and $Y_0 = \psi(Z_0)$, then $Z_0 = \psi^{-1}(Y_0)$ and $\psi|_{Z_0}$ is injective.*

   Furthermore if Y is a locally compact second countable space and ϕ is surjective, then Y_0 is open and dense in Y and $\psi|_{Z_0}$ is a homeomorphism onto Y_0.

In the next result we classify the G-equivariant factors of L/Λ, in particular we describe the factor $\phi_1 : L/\Lambda \to X$ appearing in the statement of theorem 1.2. The proof of this result uses the theorem of Ratner on orbit closures of unipotent flows and the main result of [MS].

Definition 1.1 Let Λ_1 be a closed subgroup of L. A homeomorphism τ on L/Λ_1 is called an *affine automorphism* of L/Λ_1 if there exists $\sigma \in \mathrm{Aut}(L)$ such that $\tau(gx) = \sigma(g)\tau(x)$ for all $x \in L/\Lambda_1$. The group of all affine automorphisms of L/Λ_1 is denoted by $\mathrm{Aff}(L/\Lambda_1)$. It is endowed with the compact-open topology; i.e. its open subbase consists of the sets of the form $\{\tau \in \mathrm{Aff}(L/\Lambda_1) : \tau(C) \subset U\}$, where C is a compact subset of L/Λ_1 and U is an open subset of L/Λ_1.

Remark 1.1 (1) $\mathrm{Aff}(L/\Lambda_1)$ is a locally compact topological group acting continuously on L/Λ_1. (2) If $\sigma \in \mathrm{Aut}(L)$ is such that $\sigma(\Lambda_1) = \Lambda_1$ and if $g \in L$, then the map τ on L/Λ_1 defined by $\tau(h\Lambda_1) = g\sigma(h)\Lambda_1$ for all $h \in L$ is an affine automorphism. (3) Let Λ_1' be the maximal closed normal subgroup of L contained in Λ_1. Define $\bar{L} = L/\Lambda_1'$ and $\bar{\Lambda}_1 = \Lambda_1/\Lambda_1'$. Then we have natural isomorphisms $L/\Lambda \cong \bar{L}/\bar{\Lambda}_1$ and $\mathrm{Aff}(L/\Lambda_1) = \mathrm{Aff}(\bar{L}/\bar{\Lambda}_1)$.

Theorem 1.3 *Let G be a subgroup of L which is generated by one-parameter unipotent subgroups of L contained in G. Suppose that G acts ergodically on L/Λ. Let X be a Hausdorff locally compact space with a continuous G-action and $\phi : L/\Lambda \to X$ a continuous surjective G-equivariant map. Then there exists a closed subgroup Λ_1 containing Λ, a compact group K contained in the centralizer of the subgroup of translations by elements of G in $\mathrm{Aff}(L/\Lambda_1)$, and a G-equivariant continuous surjective map $\psi : K\backslash L/\Lambda_1 \to X$ such that the following holds:*

1. *If $\rho : L/\Lambda \to K\backslash L/\Lambda_1$ is defined by $\rho(g\Lambda) = K(g\Lambda_1)$, $\forall\, g \in L$, then ρ is G-equivariant and*
$$\phi = \psi \circ \rho.$$
2. *Given a neighbourhood Ω of e in $Z_L(G)$, there exists an open dense G-invariant subset X_0 of L/Λ_1 such that for any $x \in X_0$ and $y \in L/\Lambda_1$ if $\psi(K(x)) = \psi(K(y))$ then $y \in K(\Omega x)$. In this situation, further if $\overline{Gx} = L/\Lambda_1$, then $K(y) = K(x)$.*

The above description of topological factors of unipotent flows is also of independent interest. The measurable factors of unipotent flows were described by Witte [W].

The next result is an immediate consequence of theorems 1.2 and 1.3.

Corollary 1.1 *Let L be a Lie group, Λ a lattice in L, and G a connected semisimple Lie group with finite center, realized as a closed subgroup of L. Suppose that the action of G_1 on L/Λ is minimal for any closed normal subgroup G_1 of G such that $\mathbb{R}$-rank$(G/G_1) \leq 1$. Let Y be a locally compact Hausdorff space with a continuous G-action, P a parabolic subgroup of G, and $\phi : L/\Lambda \times G/P \to Y$ a continuous surjective G-equivariant map, where G-acts diagonally on $L/\Lambda \times G/P$. Then there exist a parabolic subgroup Q of G containing P, a closed subgroup Λ_1 of L containing Λ, and a compact group K contained in the centralizer of the image of G in $\mathrm{Aff}(L/\Lambda_1)$, such that Y is G-equivariantly homeomorphic to $(K\backslash L/\Lambda_1) \times (G/Q)$ and ϕ is the natural quotient map.*

In particular if, as in question 1.1, there exists a map $\psi : Y \to L/\Lambda$ such that $\psi \circ \phi$ is the projection on the first factor, then $\Lambda_1 = \Lambda$ and K is trivial. Hence Y is G-equivariantly homeomorphic to $L/\Lambda \times G/Q$ and ϕ is the natural quotient map.

For the purpose of other applications, we obtain stronger a measure theoretic version of theorem 1.1. Before the statement, we recall some definitions.

For any Borel map $T : X \to Y$ of Borel spaces and a Borel measure λ on X, the Borel measure $T_*\lambda$ defined by $T_*\lambda(E) = \lambda(T^{-1}(E))$, for all Borel sets $E \subset Y$, is called the image of λ under T.

For any Borel measure μ on L/Λ and any $g \in L$, the translated measure $g \cdot \mu$ on L/Λ is the image of μ under the map $x \mapsto gx$ on L/Λ.

On a locally compact space X, for a sequence $\{\mu_i\}$ of finite Borel measures and μ a finite Borel measure, we say that $\mu_i \to \mu$ as $i \to \infty$, if and only if for all bounded continuous function f on X, $\int_X f\, d\mu_i \to \int_X f\, d\mu$ as $i \to \infty$.

Notation 1.1 Let G be a connected semisimple real algebraic group. Let A be an $\mathbb{R}$-split torus in G such that the set of real roots on A for the adjoint action on the Lie algebra of G forms a

root system. Fix an order on this set of roots and let Δ be the corresponding system of simple roots. Let $\overline{A}^{+}$ be the closure of the positive Weyl chamber in A. Let $\{a_i\}_{i\in\mathbb{N}}$ be a sequence in $\overline{A}^{+}$ such that for any $\alpha \in \Delta$, either $\sup_{i\in\mathbb{N}} \alpha(a_i) < \infty$ or $\alpha(a_i) \to \infty$ as $i \to \infty$. Put

$$U^{+} = \{g \in G : {a_i}^{-1} g a_i \to e \text{ as } i \to \infty\}.$$

Theorem 1.4 *Consider the notation 1.1. Assume that U^{+} is not contained in any proper closed normal subgroup of G. Suppose that G is realized as a Lie subgroup of L and that $\pi(G)$ is dense in L/Λ. Then for any probability measure λ on U^{+} which is absolutely continuous with respect to a Haar measure on U^{+},*

$$a_i \cdot \pi_*(\lambda) \to \mu_L, \qquad \textit{as } i \to \infty.$$

In other words, for any bounded continuous function f on L/Λ,

$$\lim_{i\to\infty} \int_{U^{+}} f(a_i \pi(\omega))\, d\lambda(\omega) = \int_{L/\Lambda} f\, d\mu_L.$$

In particular, for any Borel set Ω of U^{+} having strictly positive Haar measure,

$$\overline{\bigcup_{i\in\mathbb{N}} a_i \cdot \pi(\Omega)} = L/\Lambda.$$

Using this theorem we obtain the following generalization of a result due to Duke, Rudnick and Sarnak [DRS]; their result corresponds to the case of $L = G$. First we need a definition.

Let G be a semisimple Lie group. A subgroup S of G is said to be *symmetric* if there exists an involution σ of G (i.e. σ is a continuous automorphism and $\sigma^2 = 1$) such that $S = \{g \in G : \sigma(g) = g\}$. For example, any maximal compact subgroup of G is a symmetric subgroup, for it is the fixed point set of a Cartan involution of G.

Corollary 1.2 *Let G be a connected real algebraic semisimple Lie group realized as a Lie subgroup of L, S the connected component of the identity of a symmetric subgroup of G, and $\{g_i\}_{i\in\mathbb{N}}$ a sequence contained in G. Suppose that $\pi(S)$ is closed and admits an S-invariant probability measure, say μ_S. Also suppose that $\pi(G_1)$ is dense in L/Λ, for any closed normal subgroup G_1 of G such that the image of $\{g_i\}$ in $G/(SG_1)$ admits a convergent subsequence. Then the sequence of measures $g_i \cdot \mu_S$ converges to μ_L; that is, for every bounded continuous function f on L/Λ,*

$$\lim_{i\to\infty} \int_{\pi(S)} f(g_i x)\, d\mu_S(x) = \int_{L/\Lambda} f\, d\mu_L.$$

In the case of $L = G$, Eskin and McMullen [EM] gave a proof of this result using the mixing property of geodesic flows. The main technical observation in their proof is what they call 'a wave front lemma'. In the general case of $L \supset G$, our analogue of the wave front lemma is theorem 1.4.

Using the arguments of the proof of corollary 1.2, one can also deduce the following result from theorem 1.4.

Corollary 1.3 *Let G be a connected real algebraic semisimple group realized as a Lie subgroup of L. Let $\{g_i\}$ be a sequence in G. Suppose that $\pi(G_1)$ is dense in L/Λ for any closed normal subgroup G_1 of G such that the image of $\{g_i\}$ in G/G_1 admits a convergent subsequence. Then for any Borel probability measure λ on G which is absolutely continuous with respect to a Haar measure on G,*

$$g_i \pi_*(\lambda) \to \mu_L \qquad \textit{as } i \to \infty.$$

In particular, for any Borel set Ω of G having strictly positive Haar measure,

$$\overline{\bigcup_{i\in\mathbb{N}} g_i \cdot \pi(\Omega)} = L/\Lambda.$$

Using the method of our proof, one can also obtain the uniform versions of theorem 1.4 and corollary 1.2 which are similar to [EMM, Theorems 4.3-4].

The the main result of this paper is theorem 1.4 and other results (except theorem 1.3) are derived from it. The main steps of its proof are as follows. First suppose that the set of probability measures $\{a_i \cdot \pi_*(\lambda) : i \in \mathbb{N}\}$ is not is not relatively compact in the space of all probability measures on L/Λ. Using an extension of a result of Dani and Margulis [DM2], in section 2 we see that there exist a nonempty open set $\Omega \subset U^+$, a finite dimensional representation V of L, a discrete set $\{\mathbf{v}_i : i \in \mathbb{N}\} \subset V$, and a compact set $K \subset V$ such that that $a_i\Omega\cdot\mathbf{v}_i \subset K$ for infinitely many $i \in \mathbb{N}$. Via some elementary observations about representations of semisimple Lie groups, in section 5 we show that the conditions mentioned above lead to a contradiction when we restrict the representation to G. Now let a probability measure μ be a limit distribution of the sequence $\{a_i \cdot \pi_*(\lambda)\}$. We observe that μ is U^+-invariant. Using Ratner's [Ra1] description of finite measures on L/Λ which are ergodic and invariant under the action of a unipotent subgroup, in section 3 we conclude that either $\mu = \mu_L$, or μ is nonzero when restricted to the image under π of some strictly lower dimensional 'algebraic subvariety' of L. Using techniques developed in [DM1, Sh1, DM3, MS], in section 4 we see that in the later case the above type of condition on a finite dimensional representation of L must hold, and this again leads to a contradiction. Thus $\mu = \mu_L$ and hence μ_L is the only limit distribution of $\{a_i \cdot \mu_\Omega\}$.

2 A condition for returning to compact sets

In [DM2] Dani and Margulis proved that large compact sets in finite volume homogeneous spaces have relative measures close to 1 on the trajectories of unipotent flows which originate from a fixed compact set. This result was generalized in [EMS1] to a larger class of higher dimensional trajectories. In these results one considered only the case of arithmetic lattices in algebraic semisimple Lie groups defined over $\mathbb{Q}$. Here we modify them to include the case of any lattice in any Lie group.

Notation 2.1 Let G be a Lie group and $\mathfrak{g}$ the Lie algebra associated to G. For $d, m \in \mathbb{N}$, let $\mathcal{P}_{d,m}(G)$ denote the set of continuous maps $\Theta : \mathbb{R}^m \to G$ such that for all $\mathbf{c}, \mathbf{a} \in \mathbb{R}^m$ and $X \in \mathfrak{g}$, the map

$$t \in \mathbb{R} \mapsto \mathrm{Ad} \circ \Theta(t\mathbf{c} + \mathbf{a})(X) \in \mathfrak{g}$$

is a polynomial of degree at most d in each co-ordinate of $\mathfrak{g}$ (with respect to any basis).

We shall write $\mathcal{P}_d(G)$ for the set $\mathcal{P}_{d,1}(G)$.

Theorem 2.1 (Dani, Margulis) *Let G be a Lie group, Γ a lattice in G, and $\pi : G \to G/\Gamma$ the natural quotient map. Then given a compact set $C \subset G/\Gamma$, an $\epsilon > 0$, and a $d \in \mathbb{N}$, there exists a compact subset $K \subset G/\Gamma$ with the following property: For any $\Theta \in \mathcal{P}_{d,m}(G)$ and any bounded open convex set $B \subset \mathbb{R}^m$, one of the following conditions hold:*

1. $(1/\nu(B))\nu(\{\mathbf{t} \in B : \pi(\Theta(\mathbf{t})) \in K\}) \geq 1 - \epsilon$, *where ν denotes the Lebesgue measure on $\mathbb{R}^m$.*
2. $\pi(\Theta(B)) \cap C = \emptyset$.

Proof. See [Sh2, Theorem 3.1]. □

The usefulness of the above result is enhanced by the following theorem which provides an algebraic condition in place of the geometric condition $\pi(\Theta(B)) \cap C = \emptyset$.

Notation 2.2 Let G be a connected Lie group and $\mathfrak{g}$ denote the Lie algebra associated to G. Let $V_G = \oplus_{k=1}^{\dim \mathfrak{g}} \wedge^k \mathfrak{g}$, the direct sum of exterior powers of $\mathfrak{g}$, and consider the linear G-action

on V_G via the representation $\oplus_{l=1}^{\dim \mathfrak{g}} \wedge^l \mathrm{Ad}$, the direct sum of exterior powers of the adjoint representation of G on $\mathfrak{g}$.

Fix any Euclidean norm on $\mathfrak{g}$ and let $\mathcal{B} = \{\mathbf{e}_1, \ldots, \mathbf{e}_{\dim \mathfrak{g}}\}$ denote an orthonormal basis of $\mathfrak{g}$. There is a unique Euclidean norm $\|\cdot\|$ on V_G such that the associated basis of V_G given by

$$\{\mathbf{e}_{l_1} \wedge \cdots \wedge \mathbf{e}_{l_r} : 1 \leq l_1 < \ldots < l_r \leq \dim \mathfrak{g},\, r = 1, \ldots, \dim \mathfrak{g}\}$$

is orthonormal. This norm is independent of the choice of $\mathcal{B}$.

To any Lie subgroup W of G and the associated Lie subalgebra $\mathfrak{w}$ of $\mathfrak{g}$ we associate a unit-norm vector $\mathbf{p}_W \in \wedge^{\dim \mathfrak{w}} \mathfrak{w} \in V_G$.

Theorem 2.2 (Cf. [DM2]) *Let G be a connected Lie group, Γ a lattice in G, and $\pi : G \to G/\Gamma$ the natural quotient map. Let M be the smallest closed normal subgroup of G such that $\bar{G} = G/M$ is a semisimple group with trivial center and without nontrivial compact normal subgroups. Let $q : G \to \bar{G}$ be the quotient homomorphism. Then there exist finitely many closed subgroups $W_1, \ldots, W_r$ of G such that each W_i is of the form $q^{-1}(U_i)$ with U_i the unipotent radical of a maximal parabolic subgroup of $\bar{G}$, $\pi(W_i)$ is compact and the following holds: Given $d, m \in \mathbb{N}$ and reals $\alpha, \epsilon > 0$, there exists a compact set $C \subset G/\Gamma$ such that for any $\Theta \in \mathcal{P}_{d,m}(G)$, and a bounded open convex set $B \subset \mathbb{R}^m$, one of the following conditions is satisfied:*

1. *There exist $\gamma \in \Gamma$ and $i \in \{1, \ldots, r\}$ such that*

$$\sup_{\mathbf{t} \in B} \|\Theta(\mathbf{t})\gamma \cdot \mathbf{p}_{W_i}\| < \alpha.$$

2. *$\pi(\Theta(B)) \cap C \neq \emptyset$, and hence condition (1) of theorem 2.1 holds.*

Proof. Let R be the radical of G, C the maximal connected compact normal subgroup of G/R, $S = (G/R)/C$ and Z the center of S. Note that S is a semisimple Lie group without proper compact connected normal subgroups. Clearly $S/Z \cong G/M$. Therefore M is the inverse image of Z in G.

Let $H = \overline{R\Gamma}^0$. Then $H\Gamma$ is closed and $H \cap \Gamma$ is a lattice in H (see [R, Lemma 1.7]). By Auslander's theorem [R, Theorem 8.24] H is solvable, and so is its image in S. By Borel's density theorem [R, Lemma 5.4, Corollary 5.16] the image is a normal subgroup of S and therefore it has to be trivial. Hence $H \subset M^0$, and since $R \subset H$, M^0/H is compact. Since H is solvable, by Mostow's theorem [R, Theorem 3.1] $H/(H \cap \Gamma)$ is compact. Therefore $M^0/H \cap \Gamma$ is compact. So $M^0\Gamma/\Gamma$ is compact and $M^0\Gamma$ is closed.

Therefore the image Δ of Γ in S is discrete, and hence a lattice in S. Therefore by Borel's density theorem [R, Corollary 5.18] $Z\Delta$ is discrete. Hence Δ is of finite index in $Z\Delta$ and hence $M^0\Gamma$ is of finite index in $M\Gamma$. Hence $M\Gamma/\Gamma$ is compact, i.e. $\pi(M)$ is compact.

Thus $\bar{\Gamma} = q(\Gamma)$ is a lattice in $\bar{G}$ and the fibers of the map $\bar{q} : G/\Gamma \to \bar{G}/\bar{\Gamma}$ are compact M-orbits. Therefore without loss of generality, we may assume that $\bar{G} = G$.

Then there are finitely many normal connected subgroups $G_1, \dots, G_r$ of G such that $G = G_1 \times \cdots \times G_r$ and each $\Gamma_i = G_i \cap \Gamma$ is an irreducible lattice in G_i (see [R, Sect. 5.22]). In proving the theorem without loss of generality we may replace Γ by its finite-index subgroup $\Gamma_1 \times \cdots \times \Gamma_r$. In order to prove the theorem for G, it is enough to prove it for each G_i separately. Thus without loss of generality we may assume that Γ is an irreducible lattice.

The result in the case of $\mathbb{R}$-rank$(G) = 1$ can be deduced from the arguments in [D2, (2.4)].

Next suppose that $\mathbb{R}$-rank$(G) \geq 2$. Then by the arithmeticity theorem of Margulis [M2], Γ is an arithmetic lattice. Therefore there exist a semisimple algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$ and a surjective homomorphism $\rho : \mathbf{G}(\mathbb{R})^0 \to G$ with compact kernel such that, for $\Lambda = \mathbf{G}(\mathbb{Z}) \cap \mathbf{G}(\mathbb{R})^0$, the subgroup $\Gamma \cap \rho(\Lambda)$ is a subgroup of finite index in both Γ and $\rho(\Lambda)$. Again without loss of generality we may replace G by $\mathbf{G}(\mathbb{R})^0$ and Γ by Λ. In this case the result follows from [EMS1, Thm. 3.6]. $\square$

3 Description of measures invariant under a unipotent flow

In this and the next section, let G denote a Lie group, Γ a lattice in G, and $\pi : G \to G/\Gamma$ the natural quotient map.

A subgroup U of G is called *unipotent* if $\mathrm{Ad}u$ is a unipotent endomorphism of the Lie algebra of G for every $u \in U$.

Let $\mathcal{H}_\Gamma$ denote the collection of all closed connected subgroups H of G such that (1) $H \supset \Gamma$, (2) $H/H \cap \Gamma$ admits a finite H-invariant measure, and (3) the subgroup generated by all one-parameter unipotent subgroups of H acts ergodically on $H/H\cap\Gamma$ with respect to the H-invariant probability measure. In particular, the Zariski closure of $\mathrm{Ad}(H \cap \Gamma)$ contains $\mathrm{Ad}(H)$ (see [Sh1, Theorem 2.3]).

Theorem 3.1 ([Ra1, Theorem 1.1]) *The collection $\mathcal{H}_\Gamma$ is countable.*

Let W be a subgroup of G which is generated by one-parameter unipotent subgroups of G contained in W. For any $H \in \mathcal{H}_\Gamma$, define

$$\begin{aligned} N_G(H, W) &= \{g \in G : W \subset gHg^{-1}\}, \\ S_G(H, W) &= \bigcup_{\substack{H' \in \mathcal{H}_\Gamma,\, H' \subset H \\ \dim H' < \dim H}} N_G(H', W). \end{aligned}$$

Note that (see [MS, Lemma 2.4]),

$$\pi(N_G(H, W) \setminus S_G(H, W)) = \pi(N_G(H, W)) \setminus \pi(S_G(H, W)). \tag{1}$$

We reformulate Ratner's classification [Ra1] of finite measures which are invariant and ergodic under unipotent flows on homogeneous spaces of Lie groups, using the above definitions (see [MS, Theorem 2.2]).

Theorem 3.2 *Let W be a subgroup as above and μ a W-invariant probability measure on G/Γ. For every $H \in \mathcal{H}_\Gamma$, let μ_H denote the restriction of μ on $\pi(N_G(H, W) \setminus S_G(H, W))$. Then the following holds.*

1. *The measure μ_H is W-invariant, and any W-ergodic component of μ_H is of the form $g \cdot \lambda$, where $g \in N_G(H, W) \setminus S_G(H, W)$ and λ is a H-invariant measure on $H\Gamma/\Gamma$.*

2. *For any Borel measurable set $A \subset G/\Gamma$,*

$$\mu(A) = \sum_{H \in \mathcal{H}_\Gamma^*} \mu_H(A),$$

where $\mathcal{H}_\Gamma^* \subset \mathcal{H}_\Gamma$ is a countable set consisting of one representative from each Γ-conjugacy class of elements in $\mathcal{H}_\Gamma$.

In particular, if $\mu(\pi(S(G,W)) = 0$ then μ is the unique G-invariant probability measure on G/Γ.

4 Linear presentation of G-actions near singular sets

Let $C \subset \pi(N_G(H,W) \setminus S_G(H,W))$ be any compact set. It turns out that on certain neighborhoods of C in G/Γ, the G-action is equivariant with the linear G-action on certain neighbourhoods of a compact subset of a linear subspace in a finite dimensional linear G-space. We study unipotent trajectories in those thin neighbourhoods of C via this linearisation. This type of technique is developed in ([DM1, Sh1, DM3, Sh2, MS, EMS2]).

Let V_G be the representation of G as described in notation 2.2. For $H \in \mathcal{H}_\Gamma$, let $\eta_H : G \to V_G$ be the map defined by $\eta_H(g) = g\mathbf{p}_H = (\wedge^d \mathrm{Ad} g)\mathbf{p}_H$ for all $g \in G$. Let $N_G(H)$ denotes the normaliser of H in G. Define

$$N_G^1(H) = {\eta_H}^{-1}(\mathbf{p}_H) = \{g \in N_G(H) : \det(\mathrm{Ad} g|_{\mathfrak{h}}) = 1\}.$$

Proposition 4.1 ([DM3, Theorem 3.4]) *The orbit $\Gamma \cdot \mathbf{p}_H$ is closed, and hence discrete. In particular, the orbit $N_G^1(H)\Gamma/\Gamma$ is closed in G/Γ.*

Let W be a subgroup which is generated by one-parameter unipotent subgroups of G contained in W.

Proposition 4.2 ([DM3, Prop. 3.2]) *Let $V_G(H,W)$ denote the linear span of $\eta(N_G(H,W))$ in V_G. Then*

$${\eta_H}^{-1}(V_G(H,W)) = N_G(H,W).$$

Theorem 4.1 *Let $\epsilon > 0$, $d, m \in \mathbb{N}$, and a compact set $C \subset \pi(N_G(H,W) \setminus S_G(H,W))$ be given. Then there exists a compact set $D \subset V_G(H,W)$ such that given any neighbourhood Φ of D in V_G, there exists a neighbourhood Ψ of C in G/Γ such that for any $\Theta \in \mathcal{P}_{d,m}(G)$, and a bounded open convex set $B \subset \mathbb{R}^m$, one of the following conditions is satisfied:*

1. *$\Theta(B)\gamma \cdot \mathbf{p}_H \subset \Phi$ for some $\gamma \in \Gamma$.*
2. $$\frac{1}{\nu(B)}\nu(\{\mathbf{t} \in B : \Theta(\mathbf{t})\Gamma/\Gamma \in \Psi\}) < \epsilon.$$

Proof. The result is easily deduced from [Sh2, Prop. 5.4]. See also the proof of [Sh2, Thm. 5.2]. □

Some related results on unipotent flows

We recall a theorem of Ratner [Ra2] on closures of individual orbits of unipotent flows.

Theorem 4.2 (Ratner) *Let G, Γ and W be as above. Then for any $x \in G/\Gamma$, there exists a closed subgroup F of G containing W such that $\overline{Wx} = Fx$ and the orbit Fx admits a unique F-invariant probability measure, say μ_F. Also μ_F is W-ergodic.*

Next we recall a version of the main result of [MS].

Theorem 4.3 ([MS]) *Let $x \in G/\Gamma$, and sequences $\{F_i\}$ of closed subgroups of G and $g_i \to e$ in G be such that each of the orbits $F_i(g_i x)$ is closed, and admits an F_i-invariant probability measure, say μ_i. Suppose that the subgroup generated by all unipotent one-parameter subgroups of G contained in F_i acts ergodically with respect to μ_i, $\forall i \in \mathbb{N}$. Then there exits a closed subgroup F of G such that the orbit Fx is closed, and admits a F-invariant probability measure, say μ, and a subsequence of $\{\mu_i\}$ converges to μ.*

Moreover if $\mu_i \to \mu$ as $i \to \infty$, then ${g_i}^{-1} F_i g_i \subset F$ for all large $i \in \mathbb{N}$.

5 Some results on linear representations

In view of the proposition 4.1, in order to obtain further consequences when either condition 1 of theorem 2.2 or condition 1 of theorem 4.1 holds for a sequence $\{\Theta_i\} \subset \mathcal{P}_{d,m}(G)$, the following elementary result is very useful.

Linear actions of Unipotent subgroups

Lemma 5.1 *Let V be a finite dimensional real vector space equipped with a Euclidean norm. Let $\mathfrak{n}$ be a nilpotent Lie subalgebra of $\operatorname{End}(V)$. Let N be the associated unipotent subgroup of $\operatorname{Aut}(V)$. Let $\{\mathbf{b}_1, \dots, \mathbf{b}_m\}$ be a basis of $\mathfrak{n}$. Consider the map $\Theta : \mathbb{R}^m \to N$ defined as*

$$\Theta(t_1, \dots, t_m) = \exp(t_m \mathbf{b}_m) \cdots \exp(t_1 \mathbf{b}_1), \qquad \forall (t_1, \dots, t_m) \in \mathbb{R}^m.$$

For $\rho > 0$, define

$$B_\rho = \{\Theta(t_1, \dots, t_m) \in N : 0 \le t_k < \rho,\, k = 1, \dots, m\}.$$

Put

$$W = \{\mathbf{v} \in V : n \cdot \mathbf{v} = \mathbf{v},\, \forall n \in N\}.$$

Let pr_W denote the orthogonal projection on W. Then for any $\rho > 0$, there exists $c > 0$ such that for every $\mathbf{v} \in V$,

$$\|\mathbf{v}\| \le c \cdot \sup_{\mathbf{t} \in B_\rho} \|\operatorname{pr}_W(\Theta(\mathbf{t}) \cdot \mathbf{v})\|.$$

Proof. For $k = 1, \dots, m$, let $n_k \in \mathbb{N}$ be such that $\mathbf{b}_k^{n_k} = 0$. Let

$$\mathcal{I} = \{I = (i_1, \dots, i_m) : 0 \le i_k \le n_k - 1,\, k = 1, \dots, m\}.$$

For $\mathbf{t} = (t_1, \dots, t_m) \in \mathbb{R}^m$ and $I = (i_1, \dots, i_m) \in \mathcal{I}$, define

$$\mathbf{t}^I = t_m^{i_m} \cdots t_1^{i_1} \qquad \text{and} \qquad \mathbf{b}^I = \frac{\mathbf{b}_m^{i_m} \cdots \mathbf{b}_1^{i_1}}{i_m! \cdots i_1!}.$$

Then for all $\mathbf{v} \in V$ and $\mathbf{t} \in \mathbb{R}^m$, we have

$$\Theta(\mathbf{t}) \cdot \mathbf{v} = \sum_{I \in \mathcal{I}} \mathbf{t}^I \cdot (\mathbf{b}^I \mathbf{v}). \tag{2}$$

We define a transformation $T : V \to \oplus_{I \in \mathcal{I}} W$ by

$$T(\mathbf{v}) = \Big(\operatorname{pr}_W(\mathbf{b}^I \cdot \mathbf{v})\Big)_{I \in \mathcal{I}}, \qquad \forall \mathbf{v} \in V. \tag{3}$$

We claim that T is injective. To see this, suppose there exists $\mathbf{v} \in V \setminus \{0\}$ such that $T(\mathbf{v}) = 0$. Then $N \cdot \mathbf{v} \subset W^\perp$, the orthogonal complement of W. Hence $W^\perp$ contains a nontrivial N-invariant subspace. Then by Lie-Kolchin theorem, $W^\perp$ contains a nonzero vector fixed by N. Then $W \cap W^\perp \ne \{0\}$, which is a contradiction.

We consider $\oplus_{I\in\mathcal{I}}V$ equipped with a box norm; that is

$$\|(v_I)_{I\in\mathcal{I}}\| = \sup_{I\in\mathcal{I}} \|v_I\|, \qquad \text{where } v_I \in V,\ \forall I \in \mathcal{I}.$$

Since T is injective, there exists a constant $c_1 > 0$ such that

$$\|\mathbf{v}\| \le c_1 \cdot \|T(\mathbf{v})\|, \qquad \forall \mathbf{v} \in V.$$

For all $k = 1,\dots,m$, and $j_k = 1,\dots,n_k$, fix $0 < t_{k,1} < \cdots < t_{k,n_k} < \rho$ and put $M_k = \left(t_{k,j_k}^{i_k}\right)_{0\le i_k\le n_k-1,\,1\le j_k\le n_k}$ for $k = 1,\dots,m$. Then $\det M_k$ is a Vandermonde determinant and hence M_k is invertible.

Let

$$\mathcal{J} = \{J = (j_1,\dots,j_m) : 1 \le j_k \le n_k,\ k = 1,\dots,m\}.$$

For $J = (j_1,\dots,j_m) \in \mathcal{J}$, put

$$\mathbf{t}_J = (t_{1,j_1},\dots,t_{m,j_m}) \qquad \text{and} \qquad M = \left(\mathbf{t}_J^I\right)_{(I,J)\in\mathcal{I}\times\mathcal{J}}.$$

Take $\mathbf{v} \in V$. Put

$$X_{\mathcal{I}} = T(\mathbf{v}) \qquad \text{and} \qquad Y_{\mathcal{J}} = \left(\mathrm{pr}_W(\Theta(\mathbf{t}_J)\mathbf{v})\right)_{J\in\mathcal{J}}.$$

Then by equations 2 and 3,

$$M \cdot X_{\mathcal{I}} = Y_{\mathcal{J}}.$$

Since $M = M_1 \otimes \cdots \otimes M_m$ and each M_k is invertible, we have that M is invertible. Hence

$$X_{\mathcal{I}} = M^{-1} \cdot Y_{\mathcal{J}}.$$

Put $c_2 = \|M^{-1}\|$ and $c = c_1 c_2$. Then

$$\|\mathbf{v}\| \le c_1\|T(\mathbf{v})\| = c_1\|X_{\mathcal{I}}\| \le c_1c_2\|Y_{\mathcal{J}}\| = c \cdot \sup_{J\in\mathcal{J}} \|\mathrm{pr}_W(\Theta(\mathbf{t}_J)\mathbf{v})\|.$$

This completes the proof. □

Linear actions of semisimple groups

We fix the following setup for the rest of this section.

Notation 5.1 Consider the notation 1.1. Put

$$\Phi = \{\alpha \in \Delta : \alpha(a_i) \to \infty \text{ as } i \to \infty\}.$$

Let P^+ be the standard parabolic subgroup associated to the set of roots $\Delta \setminus \Phi$. Then $U^+ = \{g \in G : {a_i}^{-1} g a_i \to e \text{ as } i \to \infty\}$ is the unipotent radical of P^+. Let P^- denote the standard opposite parabolic subgroup for P^+ and let U^- be the unipotent radical of P^-. Note that

$$P^- = \{g \in G : \overline{\{a_i g {a_i}^{-1} : i \in \mathbb{N}\}} \text{ is compact}\}. \tag{4}$$

Also put $Z = P^- \cap P^+$. Then $P^- = U^- Z$. Let $\mathfrak{g}$, $\mathfrak{u}^-$, $\mathfrak{z}$, and $\mathfrak{u}^+$ denote the Lie algebras associated to G, U^-, Z, and U^+, respectively. Then

$$\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{z} \oplus \mathfrak{u}^+. \tag{5}$$

Lemma 5.2 *Consider a continuous nontrivial irreducible representation of G on a finite dimensional normed vector space V. Let $W = \{\mathbf{v} \in V : W \cdot \mathbf{v} = \mathbf{v}\}$. Let $\{\mathbf{v}_i\} \subset W$ be a sequence such that* $\inf_{i\in\mathbb{N}} \|\mathbf{v}_i\| > 0$. *Then*

$$\|a_i \cdot \mathbf{v}_i\| \to \infty \qquad \textit{as } i \to \infty.$$

Proof. Since A is $\mathbb{R}$-split, there is a finite set Λ of real characters on A such that for each $\lambda \in \Lambda$, if we define

$$V_\lambda = \{\mathbf{v} \in V : a \cdot \mathbf{v} = \lambda(a)\mathbf{v}, \, \forall a \in A\},$$

then $V = \oplus_{\lambda\in\Lambda} V_\lambda$. After passing to an appropriate subsequence, if we define

$$\begin{aligned} \Lambda_+ &= \{\lambda \in \Lambda : \lambda(a_i) \to \infty \text{ as } i \to \infty\} \\ \Lambda_- &= \{\lambda \in \Lambda : \lambda(a_i) \to 0 \text{ as } i \to \infty\}, \qquad \text{and} \\ \Lambda_0 &= \{\lambda \in \Lambda : \lambda(a_i) \to c \text{ for some } c > 0 \text{ as } i \to \infty\}, \end{aligned}$$

then $\Lambda = \Lambda_+ \cup \Lambda_0 \cup \Lambda_-$.

Since U^+ is normalized by A, we have that W is invariant under the action of A. Therefore

$$W = \oplus_{\lambda\in\Lambda}(W \cap V_\lambda).$$

Suppose that there exists $\mathbf{w} \in W \cap V_\lambda \setminus \{0\}$ for some $\lambda \in \Lambda_0 \cup \Lambda_-$. For any $g \in P^-$, we have $a_i g {a_i}^{-1} \to g_0$ for some $g_0 \in P^-$. Therefore as $i \to \infty$,

$$a_i(g\mathbf{w}) = a_i g {a_i}^{-1}(a_i\mathbf{w}) \to c(g_0\mathbf{w}) \qquad \text{for some } c \geq 0.$$

Hence $P^-\mathbf{w} \subset \oplus_{\lambda\in\Lambda_0\cup\Lambda_-} V_\lambda$. Now $U^+\mathbf{w} = \mathbf{w}$ and by notation 5.1 P^-U^+ is open in G. Therefore $G \cdot \mathbf{w} \subset \oplus_{\lambda\in\Lambda_0\cup\Lambda_-} V_\lambda$. Since V is irreducible, $\Lambda = \Lambda_0 \cup \Lambda_-$. Now since G is semisimple, $\det g = 1$ for all $g \in G$ and hence $\Lambda_- = \emptyset$. Thus $\Lambda = \Lambda_0$.

Now for any relatively compact neighbourhood Ω of U^+ and any $\mathbf{v} \in V_\lambda$, there exists a compact ball $B \subset V$ such that for all $i \in \mathbb{N}$,

$$B \supset a_i\Omega \cdot \mathbf{v} = (a_i\Omega {a_i}^{-1})a_i \cdot \mathbf{v} = \lambda(a_i)(a_i\Omega {a_i}^{-1})\mathbf{v}.$$

Since $\lambda(a_i) \to c$ for some $c > 0$ and $\cup_{i\in\mathbb{N}} a_i\Omega {a_i}^{-1} = U^+$, we have $U^+ \cdot \mathbf{v} \subset c^{-1}B$. Since U^+ acts on V by unipotent linear transformations, we obtain that $U^+ \cdot \mathbf{v} = \mathbf{v}$. Thus U^+ acts trivially on V. Since the kernel of G action on V is a normal subgroup of G containing U^+, it is equal to G by our assumption. This contradicts our hypothesis in the lemma that the action of G is nontrivial. This proves that $W \subset \sum_{\lambda\in\Lambda_+} V_\lambda$, and the conclusion of the lemma follows. $\square$

Corollary 5.1 *Consider a continuous representation of G on a finite dimensional vector space V with a Euclidean norm. Let $L = \{\mathbf{v} \in V : G \cdot \mathbf{v} = \mathbf{v}\}$. Let $\{\mathbf{v}_i\}$ be a discrete subset of V contained in $V \setminus L$. Then for any nonempty open set $\Omega \subset U^+$,*

$$\sup_{\omega\in\Omega} \|a_i\omega \cdot \mathbf{v}_i\| \to \infty \qquad \textit{as } i \to \infty. \tag{6}$$

Proof. Let L' be the sum of all G-invariant irreducible subspaces of $\dim \geq 2$. After passing to a subsequence, one of the following holds:

$$\text{(A)} \quad \|\mathrm{pr}_L(\mathbf{v}_i)\| \to \infty, \qquad \text{or} \qquad \text{(B)} \quad \inf_{i\in\mathbb{N}} \|\mathrm{pr}_{L'}(\mathbf{v}_i)\| > 0.$$

If (A) holds then equation 6 is obvious. If (B) holds, then there exists an irreducible G-subspace $V_1 \subset L'$ such that $\inf_{i\in\mathbb{N}} \|\mathrm{pr}_{V_1}(\mathbf{v}_i)\| > 0$. Therefore, without loss of generality, by replacing $\{\mathbf{v}_i\}$

by $\{\mathrm{pr}_{V_1}(\mathbf{v}_i)\}$ and V by V_1 we may assume that G acts nontrivially and irreducibly on V and $\inf_{i\in\mathbb{N}}\|\mathbf{v}_i\|>0$.

Let $\omega_0\in\Omega$. Then $\inf_{i\in\mathbb{N}}\|\omega_0\mathbf{v}_i\|>0$. Therefore replacing $\{\mathbf{v}_i\}$ by $\{\omega_0\mathbf{v}_i\}$ and Ω by $\Omega{\omega_0}^{-1}$, we may assume that $e\in\Omega$.

Let $W=\{\mathbf{v}\in V:U^+\cdot\mathbf{v}=\mathbf{v}\}$. By lemma 5.1, there exists $c>0$ such that for all $i\in\mathbb{N}$,

$$\sup_{\omega\in\Omega}\|\mathrm{pr}_W(\omega\cdot\mathbf{v}_i)\|\geq c\|\mathbf{v}_i\|\geq c\cdot\inf_{j\in\mathbb{N}}\|\mathbf{v}_j\|.$$

Since $\inf_{j\in\mathbb{N}}\|\mathbf{v}_j\|>0$, by lemma 5.2,

$$\sup_{\omega\in\Omega}\|a_i\cdot\omega\mathbf{v}_i\|\geq\sup_{\omega\in\Omega}\|a_i\cdot\mathrm{pr}_W(\omega\cdot\mathbf{v}_i)\|\to\infty\qquad\text{as } i\to\infty.$$

□

6 Proofs of the main results

Translates of horospherical patches

Proof of theorem 1.4.

Since U^+ is σ-compact, without loss of generality we may assume that $\mathrm{supp}(\lambda)$ is compact. Let $\mathfrak{u}^+$ denote the Lie algebra of U^+. We identify $\mathfrak{u}^+$ with $\mathbb{R}^m$ ($m=\dim\mathfrak{u}^+$). Let B be a ball in $\mathfrak{u}^+$ around the origin such that $\mathrm{supp}(\lambda)\subset\exp(B)$. Let ν be the restriction of the Lebesgue measure on B. By our hypothesis, λ is absolutely continuous with respect to $\exp_*(\nu)$, denoted by $\lambda\ll\exp_*(\nu)$.

For each $i\in\mathbb{N}$, define $\Theta_i:\mathbb{R}^m\to G\subset L$ as $\Theta_i(\mathbf{t})=a_i\exp(\mathbf{t})$, $\forall\mathbf{t}\in\mathbb{R}^m\cong\mathfrak{u}^+$. Since $\mathfrak{u}^+$ is a nilpotent Lie algebra, there exists $d\in\mathbb{N}$ such that $\Theta_i\in\mathcal{P}_{d,m}(L)$, $\forall i\in\mathbb{N}$.

Claim 6.1 *Given $\delta>0$ there exists a compact set $K\subset L/\Lambda$ such that*

$$(a_i\pi_*(\lambda))(K)>1-\delta,\qquad\forall i\in\mathbb{N}.$$

Suppose that the claim fails to hold. Since $\lambda\ll\exp_*(\nu)$, there exists an $\epsilon>0$ such that for any compact set $K\subset L/\Lambda$,

$$\frac{1}{\nu(B)}(\Theta_i)_*(\nu)(K)<1-\epsilon,\qquad\text{for } i \text{ in a subsequence.}$$

We apply theorems 2.1 and 2.2 for the Lie group L, the lattice Λ, and the polynomial maps $\Theta_i\in\mathcal{P}_{d,m}(L)$, $\forall i\in\mathbb{N}$. Then by passing to a subsequence, there exists a continuous representation of L on a finite dimensional vector space V with a Euclidean norm and a nonzero vector $\mathbf{p}\in V$ such that the following holds: (1) the orbit $\Gamma\cdot\mathbf{p}$ is discrete (see proposition 4.1), and (2) for each $i\in\mathbb{N}$ there exists $\mathbf{v}_i\in\Gamma\cdot\mathbf{p}$ such that

$$\sup_{\omega\in\exp(B)}\|a_i\omega\cdot\mathbf{v}_i\|\to 0\qquad\text{as } i\to\infty.\tag{7}$$

After passing to a subsequence, we may assume that $G\cdot\mathbf{v}_i\neq\mathbf{v}_i$, $\forall i\in\mathbb{N}$. Then corollary 5.1 contradicts equation 7. This proves the claim.

By claim 6.1, after passing to a subsequence, we may assume that the sequence $a_i\cdot\pi_*(\lambda)\to\mu$ as $i\to\infty$, where μ is a probability measure on L/Λ.

Claim 6.2 *The measure μ is U^+-invariant.*

To prove the claim, let $u \in U^+$. Then for all $i \in \mathbb{N}$,

$$u(a_i\pi_*(\lambda)) = a_i(u_i\pi_*(\lambda)) = a_i\pi_*(u_i\lambda), \tag{8}$$

where $u_i = {a_i}^{-1}ua_i \in U^+$. Note that $u_i \to e$ as $i \to \infty$.

Let η be a Haar measure on U^+. Since $\lambda \ll \eta$, there exists $h \in L^1(U, \eta)$ such that $d\lambda = h\,d\eta$. Now for any bounded continuous function f on L/Λ,

$$\begin{array}{rl} & |\int f\,d[a_i\pi_*(u_i\lambda))] - \int f\,d[a_i\pi_*(\lambda)]| \\ = & |\int_{U^+} f(a_i\pi(u_i\omega))\,d\lambda(\omega) - \int_{U^+} f(a_i\pi(\omega))\,d\lambda(\omega)| \\ = & |\int_{U^+} f(a_i\pi(u_i\omega)h(\omega)\,d\eta(\omega) - \int_{U^+} f(a_i\pi(\omega))h(\omega)\,d\eta(\omega)| \\ = & \left|\int_{U^+} f(a_i\pi(\omega))h({u_i}^{-1}\omega)\,d\eta(\omega) - \int_{U^+} f(a_i\pi(\omega))h(\omega)\,d\eta(\omega)\right| \\ \leq & \sup|f| \cdot \int_{U^+} |h({u_i}^{-1}\omega) - h(\omega)|\,d\eta(\omega) \to 0 \qquad \text{as } i \to \infty, \end{array} \tag{9}$$

because the left regular representation of of U^+ on $L^1(U^+, \eta)$ is continuous.

Since $a_i\pi_*(\lambda) \to \mu$ as $i \to \infty$, by equation 9, we get $a_i\pi_*(u_i\lambda)) \to \mu$ as $i \to \infty$. Therefore by equation 8, $u\mu = \mu$. This completes the proof of the claim.

In view of claim 6.2, we apply theorem 3.2 to $W = U^+$. Then there exists a closed subgroup H of L in the collection $\mathcal{H}_\Lambda$, such that

$$\mu(\pi(S_L(H, U^+)) = 0 \qquad \text{and} \qquad \mu(\pi(N_L(H, U^+))) > 0.$$

Let a compact set $C \subset \pi(N_L(H, U^+))\setminus\pi(S_L(H, U^+))$ be such that $\mu(C) > 0$. Since $\lambda \ll \exp_*(\nu)$, there exists $\epsilon > 0$ such that for any Borel measurable set $E \subset U^+$,

$$\frac{1}{\nu(B)}\exp_*(\nu)(E) < \epsilon \Rightarrow \lambda(E) < \mu(C)/2. \tag{10}$$

Let the finite dimensional vector space V_L and the unit vector $\mathbf{p}_H \in V_L$ be as described in notation 2.2, for L in place of G there. We apply theorem 4.1 for $\epsilon > 0$, $d \in \mathbb{N}$, and $m \in \mathbb{N}$ chosen as above, and the compact set $C \subset \pi(N_L(H, U^+)) \setminus \pi(S_L(H, U^+))$ as above. Then there exists a relatively compact set $\Phi \subset V_L$ and an open neighbourhood Ψ of C in L/Λ such that for each $i \in \mathbb{N}$, applying the theorem to Θ_i in place of Θ, one of the following conditions holds:

1. There exists $\mathbf{v}_i \in \Lambda \cdot \mathbf{p}_H$ such that
$$a_i \exp(B) \cdot \mathbf{v}_i \subset \Phi.$$

2. $$\frac{1}{\nu(B)}\nu(\{\mathbf{t} \in B : \pi(a_i \exp(\mathbf{t})) \in \Psi\}) < \epsilon.$$

Since $a_i\pi_*(\lambda) \to \mu$ as $i \to \infty$ and Ψ is a neighbourhood of C, there exists $i_0 \in \mathbb{N}$ such that $\lambda(\pi^{-1}({a_i}^{-1}\Psi) \cap U^+) > \mu(C)/2$ for all $i \geq i_0$. Therefore by equation 10, condition 1 must hold for all $i \geq i_0$. Now by passing to a subsequence, there exists $\mathbf{v}_i \in \Lambda \cdot \mathbf{p}_H$ for each $i \in \mathbb{N}$ such that

$$a_i \exp(B) \cdot \mathbf{v}_i \subset \Phi. \tag{11}$$

By proposition 4.1, the sequence $\{\mathbf{v}_i\}$ is discrete. By corollary 5.1 and equation 11, there exists $i_0 \in \mathbb{N}$ such that $G \cdot \mathbf{v}_{i_0} = \mathbf{v}_{i_0}$. Let $\gamma \in \Lambda$ such that $\mathbf{v}_{i_0} = \gamma\mathbf{p}_H$. Then

$$G \cdot \gamma \cdot \mathbf{p}_H = \gamma \cdot \mathbf{p}_H.$$

Thus $G \subset \gamma N^1_L(H)\gamma^{-1}$. But $\pi(N^1_L(H))$ is closed in L/Λ by proposition 4.1, and $\pi(G)$ is dense in L/Λ. Therefore we conclude that H is a normal subgroup of L. Since $N_L(H, U^+) \supset C \neq \emptyset$, this implies in particular that U^+ is contained in H. Thus $U^+ \subset G \cap H$ and $G \cap H$ is normal in G. Therefore by our hypothesis $G \cap H = G$, or in other words $G \subset H$. Again since $\pi(G)$ is dense in L/Λ, we have $H = L$. Therefore $\mu(\pi(S(L, U^+))) = 0$. Hence by theorem 3.2, we have that μ is L-invariant. This completes the proof of the theorem. $\square$

Translates of orbits of symmetric subgroups

First we make some observations. For the results stated below, let (U,ν_1) and (V,ν_2) be locally compact second countable spaces with Borel measures.

Proposition 6.1 *Let λ be a Borel probability measure on $U\times V$ which is absolutely continuous with respect to $\nu_1\times\nu_2$, denoted by $\lambda\ll\nu_1\times\nu_2$. Then there exists a probability measure $\lambda_1\ll\nu_1$ on U, and for almost all $u\in(U,\lambda_1)$, there exists a probability measure $\lambda_u\ll\delta_u\times\nu_2$ on $\{u\}\times V$, where δ_u is the point mass at $\{u\}$, such that the following holds: For any bounded continuous function f on $U\times V$, the map $u\mapsto\int_{\{u\}\times V} f\,d\lambda_u$ is λ_1-measurable, and*

$$\int_{U\times V} f\,d\lambda=\int_U\left(\int_{\{u\}\times V} f\,d\lambda_u\right)d\lambda_1(u).$$

Proof. Let $h=d\lambda/d(\nu_1\times\nu_2)\geq 0$ be the Radon-Nikodym derivative. For any $u\in U$, put $\alpha(u)=\int_V h(u,v)\,d\nu_2(v)$. Let $C=\{u\in U:\alpha(u)>0\}$. Let λ_1 be the restriction of ν_1 to C. For almost any $u\in(U,\lambda_1)$, let λ_u be the Borel measure on $\{u\}\times V$ such that $d\lambda_u/d[\delta_u\times\nu_2]=h(u,\cdot)/\alpha(u)$. Now the conclusion of the proposition follows from Fubini's theorem. □

For the propositions stated below, let G be a locally compact topological group acting continuously on a locally compact space X. Let $\{a_i\}$ be a sequence in G and μ a Borel probability measure on X.

Proposition 6.2 *Let λ be a probability measure on X such that $a_i\lambda\to\mu$ as $i\to\infty$. Let $b\in G$ such that $\overline{\{a_iba_i{}^{-1}:i\in\mathbb{N}\}}$ is compact. If μ is G-invariant, then $a_i(b\lambda)\to\mu$ as $i\to\infty$.*

Proof. First observe that there is no loss of generality in passing to a subsequence. Therefore we may assume that $a_iba_i{}^{-1}\to g$ for some $g\in G$. Now

$$a_i(b\lambda)=(a_iba_i{}^{-1})(a_i\lambda)\to g\mu\qquad\text{as } i\to\infty.$$

Since $g\mu=\mu$, the proof is complete. □

For the next two propositions, assume that G contains the spaces U and V. Fix $x_0\in X$, and let $\rho:U\times V\to X$ be the map given by $\rho(u,v)=uvx_0$, $\forall(u,v)\in U\times V$.

Proposition 6.3 *Let the notation be as in proposition 6.1. Suppose that for almost all $u\in(U,\lambda_1)$, we have $a_i\rho_*(\lambda_u)\to\mu$ as $i\to\infty$. Then $a_i\rho_*(\lambda)\to\mu$ as $i\to\infty$.*

Proof. Let f be bounded continuous function on X. Then

$$\begin{array}{rcl}\int_X f\,d[a_i\rho_*(\lambda)] & = & \int_{U\times V} f(a_i\rho(\omega))\,d\lambda(\omega)\\ & = & \int_V d\lambda_1(u)\cdot\int_{\{u\}\times V} f(a_i\rho(\omega))\,d\lambda_u(\omega)\\ & = & \int_V d\lambda_1(u)\cdot\int_X f\,d[a_i\rho_*(\lambda_u)]\\ & \to & \int_V d\lambda_1(u)\cdot\int_X f\,d\mu\qquad\text{as } i\to\infty\\ & = & \int_X f\,d\mu.\end{array}$$

□

By similar arguments we obtain the following result.

Proposition 6.4 *Suppose that $a_iua_i{}^{-1}\to e$ as $i\to\infty$ for all $u\in U$. Then as $i\to\infty$,*

$$a_i\rho_*(\nu_2)\to\mu\Leftrightarrow a_i\rho_*(\nu_1\times\nu_2)\to\mu.$$

Proof of corollary 1.2.

Using the results in [S, Section 7.1] there exist an $\mathbb{R}$-split torus $A \subset G$ and a maximal compact subgroup K of G such that the following holds: (1) $\sigma(a) = a^{-1}$, $\forall a \in A$, (2) the set of real roots of A for the adjoint action on the Lie algebra of G forms a root system, and (3) G admits a decomposition $G = K\overline{A}^{+}S$, were $\overline{A}^{+}$ denotes the closure of the positive Weyl chamber with respect to a system Δ of simple roots on A.

Using this decomposition and by passing to a subsequence, without loss of generality we may assume the following: (1) $g_i = a_i \in \overline{A}^{+}$ for all $i \in \mathbb{N}$; (2) $\{a_i\}_{i\in\mathbb{N}}$ has no convergent subsequence, (because otherwise $G_1 = \{e\}$ and $\pi(e)$ cannot be dense in L/Λ); and (3) for any $\alpha \in \Delta$, either $\sup_{i\in\mathbb{N}} \alpha(a_i) < \infty$ or $\alpha(a_i) \to \infty$ as $i \to \infty$.

For the rest of the proof, consider the notation 5.1.

Let G_1 be the smallest closed normal subgroup of G containing U^+. Then it is straightforward to verify that the projection of $\{a_i\}$ on G/G_1 is relatively compact. Therefore by our hypothesis, $\overline{\pi(G_1)} = L/\Lambda$.

Take any $g_0 \in S$ and define $\rho(h) = \pi(hg_0)$ for all $h \in L$. Since any closed connected normal subgroup of G_1 is also normal in G, we can apply theorem 1.4 to G_1 in place of G and ρ in place of π. Then for any probability measure ν on U^+ which is absolutely continuous with respect to a Haar measure on U^+, we have

$$a_i\rho_*(\nu) \to \mu_L, \qquad \text{as } i \to \infty. \tag{12}$$

Since $\sigma(a) = a^{-1}$ ($\forall a \in A$), for any $X \in \mathfrak{u}^+$, we have $\sigma(X) \in \mathfrak{u}^-$ and $X + \sigma(X) \in \mathfrak{s}$. Also $\sigma(\mathfrak{z}) = \mathfrak{z}$. Now by equation 5,

$$\mathfrak{u}^- \oplus \mathfrak{s} = \mathfrak{u}^- \oplus (\mathfrak{s} \cap \mathfrak{z}) \oplus \mathfrak{u}^+. \tag{13}$$

Then by implicit function theorem, there exist relatively compact neighbourhoods Ω^-, Ω^0, Ω^+ and Φ of e in U^-, $(Z \cap S)U^-$, U^+ and S, respectively, such that for any open set Ψ of Φ, we have that $\Omega^-\Psi$ is an open subset of $\Omega^0\Omega^+$. Also we may assume that under the multiplication map $\Omega^- \times \Phi \cong \Omega^-\Phi$ and $\Omega^0 \times \Omega^+ \cong \Omega^0\Omega^+$.

Let ν_- and ν' be probability measures obtained by restricting Haar measures of U^- and S to Ω^- and Ψ, respectively. Then $\lambda = \nu_- \times \nu'$ is a smooth measure on $\Omega^- \times \Psi$. By choosing Ψ small enough, we can ensure that $\rho_*(\nu')$ is a multiple of μ_S restricted to $\rho(\Psi)$. Since $g_0 \in S$ chosen in the definition of ρ is arbitrary and since there is enough flexibility in the choices of Φ and Ψ, to prove that $a_i\mu_S \to \mu_L$, it is enough to show that $a_i\rho_*(\nu') \to \mu_L$ as $i \to \infty$.

By proposition 6.4, as $i \to \infty$, $a_i\rho_*(\nu') \to \mu_L$ if and only if $a_i\rho_*(\lambda) \to \mu_L$. Therefore to complete the proof of the corollary, it is enough to show the following.

Claim 6.3 *As $i \to \infty$, $a_i\rho_*(\lambda) \to \mu_L$.*

Since $\Omega^-\Psi \subset \Omega^0\Omega^+$, λ can be treated as a measure on $\Omega^0 \times \Omega^+$. Let ν_1 and ν_2 be the probability measures obtained by restricting the Haar measures on $(Z \cap S)U^-$ and U^+ to Ω^0 and Ω^+, respectively. Since λ is a smooth measure, $\lambda \ll \nu_1 \times \nu_2$ (see equation 13). Decompose λ as in proposition 6.1. Then for almost all $\omega \in (\Omega^0, \lambda_1)$, we have $\lambda_\omega \ll \omega\nu_2$. Put $\nu_\omega = \omega^{-1}\lambda_\omega$. Then $\nu_\omega \ll \nu_2$. Hence by equation 4, equation 12, and proposition 6.2,

$$a_i\rho_*(\lambda_\omega) = a_i(\omega\rho_*(\nu_\omega)) \to \mu_L \qquad \text{as } i \to \infty.$$

Now by proposition 6.3, $a_i\pi_*(\lambda) \to \mu_L$ as $i \to \infty$. This completes the proof of the claim, and also the proof of the corollary. $\square$

Continuous G-equivariant factors of $G/P \times L/\Lambda$

First we recall the following result from [D3, Section 2].

Proposition 6.5 (Dani) *Let G be a semisimple group with finite center and $\mathbb{R}\text{-rank}(G) \geq 2$. Let P be a parabolic subgroup of G. Then given $g \in G \setminus P$, there exist $k \in \mathbb{N}$ ($k < \mathbb{R}\text{-rank}(G)$), elements $g_1, \dots, g_{k+1}$ in G, and one-parameter unipotent subgroups $\{u_1(t)\}, \dots, \{u_k(t)\}$ of G contained in P such that the following holds:*

1. *$g_1 = g$, $g_k \notin P$, and $g_{k+1} = e$.*
2. *For each $i = 1, \dots, k$,*
$$u_i(t)g_iP \to g_{i+1}P \text{ in } G/P \text{ as } t \to \infty.$$
3. *There exists a semisimple element a of G in $g_kPg_k{}^{-1} \cap P$ such that if U^+ is the associated horospherical subgroup then $U^+ \subset g_kPg_k{}^{-1} \cap P$, and if G_1 denotes the smallest normal subgroup of G containing U^+, then $\mathbb{R}\text{-rank}(G/G_1) \leq 1$.*

Proof. Apply [D3, Corollary 2.3] iteratively. Also use the proofs of [D3, Corollary 2.6 and Lemma 2.7]. □

Now we obtain the analogue of [D3, Lemma 1.4] by using theorem 1.1 in place of [D3, Lemma 1.1]. Also we use the recurrence conclusion of theorem 4.2 of Ratner in place of [D3, Lemma 1.6].

Proposition 6.6 *Let the notation and assumptions be as in theorem 1.2. Let $x, y \in L/\Lambda$ and $g \in G \setminus P$. If $\phi(x, gP) = \phi(y, P)$, then there exists a parabolic subgroup Q containing $\{g\} \cup P$ such that $\phi(z, P) = \phi(z, qP)$ for all $z \in \overline{Gx}$ and $q \in Q$. Moreover, $\phi(y, P) = \phi(y, qP)$ for all $q \in Q$.*

Proof. Let $k \in \mathbb{N}$, elements $g_1, \dots, g_{k+1}$ in G, the one-parameter unipotent subgroups $\{u_i(t)\}$ contained in P, and a semisimple element a of G and the associated expanding horospherical subgroup U^+ be as in proposition 6.5. For each $i = 1, \dots, k$, by Ratner's theorem 4.2 applied to the diagonal action of $\{u_i(t)\}$ on $L/\Lambda \times L/\Lambda$, there exists a sequence $t_n \to \infty$ such that $(u_i(t_n)x, u_i(t_n)y) \to (x, y)$ as $n \to \infty$. Now for any $i \in \{1, \dots, k\}$,

$$\phi(x, g_iP) = \phi(y, P) \Rightarrow \phi(u_i(t_n)x, u_i(t_n)g_iP) = \phi(u_i(t_n)y, P), \forall n \in \mathbb{N}.$$

In the limit as $n \to \infty$, we get $\phi(x, g_{i+1}P) = \phi(y, P)$. Since $g_1 = g$, by induction on i, we get that $\phi(x, g_iP) = \phi(y, P)$ for all $1 \leq i \leq k + 1$.

In particular, since $g_{k+1} = e$,

$$\phi(x, g_kP) = \phi(y, P) = \phi(x, P).$$

Since $F = \{a^n : n \in \mathbb{N}\} \cdot U^+ \subset g_kPg_k{}^{-1} \cap P$, we have that

$$\phi(fx, g_kP) = \phi(fx, P), \, \forall f \in F.$$

Let G_1 be the smallest closed normal subgroup of G containing U^+. Then by the choice of a as in Proposition 6.5, $\mathbb{R}\text{-rank}(G/G_1) \leq 1$. Therefore by the hypothesis in theorem 1.2, $\overline{G_1x} = \overline{Gx}$. By theorem 4.2, $\overline{Gx}$ is an orbit of a closed subgroup, say L', of L containing G, and the stabilizer of x in L', say Λ', is a lattice in L'. Applying theorem 1.1 to L' and Λ' in place of L and Λ, respectively, we conclude that $\overline{Fx} = \overline{Gx}$. Thus

$$\phi(z, g_kP) = \phi(z, P), \qquad \forall z \in \overline{G_1x} = \overline{Gx}.$$

Put

$$Q = \{h \in G : \phi(z, fhP) = \phi(z, fP), \ \forall z \in \overline{Gx} \text{ and } \forall f \in G\}. \tag{14}$$

Then Q is a closed subgroup of G containing $P \cup \{g_k\}$. Since $g_k \notin P$,

$$Q \neq P. \tag{15}$$

Now if $g \notin Q$, then replacing P by Q and L/Λ by $\overline{Gx}$, we repeat the whole argument. Note that by definition the new set given by equation 14 still turns out to be same as Q. This fact contradicts the new equation 15. This completes the proof. □

Proof of theorem 1.2.

Define the equivalence relation

$$R = \{(x, y) \in L/\Lambda \times L/\Lambda : \phi(x, gP) = \phi(y, gP) \text{ for some } g \in G\}$$

on L/Λ. Clearly R is a closed subset of $L/\Lambda \times L/\Lambda$ invariant under the diagonal action of G. Let X be the space of equivalence classes of R and let $\phi_1 : L/\Lambda \to X$ be the map taking any element of L/Λ to its equivalence class. Equip X with the quotient topology. Then X is a locally compact Hausdorff space.

For any $x \in L/\Lambda$, put

$$\mathcal{Q}(x) = \{h \in G : \phi(x, gP) = \phi(x, ghP), \ \forall g \in G\}.$$

Observe that $\mathcal{Q}(x)$ is a closed subgroup of G containing P and for any $y \in \overline{Gx}$, we have $\mathcal{Q}(y) \supset \mathcal{Q}(x)$. Let $x_0 \in L/\Lambda$ such that $\overline{Gx_0} = L/\Lambda$ and put $Q = \mathcal{Q}(x_0)$. Then $\mathcal{Q}(y) \supset Q$ for all $y \in L/\Lambda$. Since Q is a parabolic subgroup of G, there are only finitely many closed subgroups of G containing Q. Therefore the set $X_Q := \{x \in L/\Lambda : \mathcal{Q}(x) = Q\}$ is open in L/Λ. Also X_Q is nonempty and G-invariant. Now since G acts ergodically on L/Λ, the set $L/\Lambda \setminus X_Q$ is closed and nowhere dense.

Note that for any $x, y \in L/\Lambda$, if $\phi_1(x) = \phi_1(y)$ then by proposition 6.6, we have that $\mathcal{Q}(x) = \mathcal{Q}(y)$. Let $\rho : L/\Lambda \times G/P \to X \times G/Q$ be the (G-equivariant) map defined by $\rho(x, gP) = (\phi_1(x), gQ)$ for all $x \in L/\Lambda$ and $g \in G$. Then there exists a uniquely defined map $\psi : X \times G/Q \to Y$ such that $\phi = \psi \circ \rho$. It is straightforward to verify that ψ is continuous and G-equivariant.

Take any $x \in X_Q$, $y \in L/\Lambda$, and $g, h \in G$ such that $\phi(x, ghP) = \phi(y, gP)$. Then $\phi_1(y) = \phi_1(x)$, and hence $h \in \mathcal{Q}(y) = \mathcal{Q}(x) = Q$. This proves that ψ restricted to $\phi_1(X_Q) \times G/Q$ is injective and $y \in X_Q$.

Now if Y is a locally compact second countable space and ϕ is surjective, then using Baire's category theorem for Hausdorff locally compact second countable spaces, one can show that ϕ is an open map. This completes the proof of the theorem. □

Continuous G-equivariant factors of L/Λ

Proof of theorem 1.3.

Define $\Lambda_1 = \{h \in L : \phi(gh\Lambda) = \phi(g\Lambda), \forall g \in L\}$. Then Λ_1 is a closed subgroup of L containing Λ. Since G-acts ergodically on L/Λ, $\mathrm{Ad}(\Lambda)$ is Zariski dense in $\mathrm{Ad}(L)$ (see [Sh1, Theorem 2.3]). Therefore Λ_1^0 is a normal subgroup of L. Let Λ_1' be the largest subgroup of Λ_1 which is normal in L. In view of 1.1 (3), replacing L by L/Λ_1', Λ by Λ_1/Λ_1', and G by its image in L/Λ_1', without loss of generality we may assume that $\Lambda_1' = \{e\}$ and $\Lambda_1 = \Lambda$.

Define the equivalence relation

$$R = \{(x, y) \in L/\Lambda \times L/\Lambda : \phi(x) = \phi(y)\}$$

on L/Λ. Then R is closed and $\Delta(G)$-invariant, where $\Delta : L \to L \times L$ denotes the diagonal embedding of L in $L \times L$.

Let

$$K = \{\tau \in \mathrm{Aff}(L/\Lambda) : (z, \tau(z)) \in R \text{ and } \tau(gz) = g\tau(z),\ \forall z \in L/\Lambda,\ \forall g \in G\}$$

and

$$X_1 = \{x \in L/\Lambda : \overline{Gx} = L/\Lambda\}.$$

Note that $X_1 \neq \emptyset$, since G acts ergodically on L/Λ.

Claim 6.4 *Let $(x, y) \in R$. If $x \in X_1$, then $y \in X_1$ and there exists $\tau \in K$ such that $y = \tau(x)$.*

The claim is proved as follows. Since $\Delta(G)$ is generated by one-parameter unipotent subgroups of $L \times L$, by Ratner's theorem 4.2 there exists a closed subgroup F of $L \times L$ containing $\Delta(G)$ such that

$$\overline{\Delta(G) \cdot (x, y)} = F \cdot (x, y)$$

and $F \cdot (x, y)$ admits an F-invariant probability measure, say λ.

Let $p_i : L \times L \to L$ denote the projection on the i-th coordinate, where $i = 1, 2$. Then $(\pi \circ p_1)_*(\lambda)$ is a $p_1(F)$-invariant probability measure on $p_1(F)x$. Hence the orbit $p_1(F)x$ is closed (see [R, theorem 1.13]). Since $G \subset p_1(F)$ and $\overline{Gx} = L/\Lambda$, we have that $p_1(F) = L$. Let $N_1 = p_1(F \cap \ker(p_2))$. Then N_1 is a normal subgroup of $p_1(F) = L$ and $(N_1 z, w) \subset R$ for all $(z, w) \in F \cdot (x, y)$. Therefore $N_1 \subset \Lambda_1' = \{e\}$. Thus $F \cap \ker(p_2) = N_1 \times \{e\} = \{e\}$. Now since $p_1(F) = L$ and $p_2|_F$ is injective, $\dim(p_2(F)) = \dim(L)$. Since L is connected, $p_2(F) = L$. Thus $p_2|_F$ is an isomorphism.

Now $\overline{Gy} \supset p_2(F)y = L/\Lambda$. Hence $y \in X_1$. Now interchanging the roles of x and y in the above argument, we conclude that $p_1|_F$ is an isomorphism. Let $\sigma = p_2 \circ (p_1|_F)^{-1}$. Then $\sigma \in \mathrm{Aut}(L)$ and

$$F = \{(g, \sigma(g)) \in L \times L : g \in L\}.$$

Thus $(gx, \sigma(g)y) \in R$ for all $g \in L$. Now for any $\delta \in L$, if $\delta x = x$, then $(gx, \sigma(g)\sigma(\delta)y) \in R$ for all $g \in L$. Let $h \in L$ such that $y = h\Lambda$. Then $\phi(\sigma(g)h\Lambda) = \phi(\sigma(g)\sigma(\delta)h\Lambda)$ for all $g \in L$. Since $\sigma(L)h = L$, we conclude that $h^{-1}\sigma(\delta)h \in \Lambda_1$. Now since $\Lambda_1 = \Lambda$, we have that $\sigma(\delta)y = y$. Therefore the map $\tau : L/\Lambda \to L/\Lambda$, given by $\tau(gx) = \sigma(g)y$ for all $g \in L$, is well defined and $\tau \in \mathrm{Aff}(L/\Lambda)$.

Therefore

$$F \cdot (x, y) = \{(z, \tau(z)) : z \in L/\Lambda\}.$$

Since $\Delta(G) \subset F$, we have that $\sigma(g) = g$ and hence $\tau(gz) = gz$, for all $g \in G$. Thus $\tau \in K$, and the proof of the claim is complete.

Claim 6.5 *The group K is compact.*

We prove the claim as follows. Clearly, K is a closed subset of $\mathrm{Aff}(L/\Lambda)$, and hence it is locally compact. Let μ_L denote the L-invariant probability measure on L/Λ. Then $\mu_L(X_1) = 1$. For any $x \in X_1$, if $y \in \overline{K \cdot x}$ then $(x, y) \in R$, and by claim 6.4 there exists $\tau \in K$ such that $y = \tau(x)$. Thus $K \cdot x$ is closed for all $x \in X_1$. Therefore by Hedlund's Lemma and the ergodic decomposition of μ_L with respect to the action of K on L/Λ, we have that almost all K-ergodic components are supported on closed K-orbits. Thus for almost all $x \in L/\Lambda$, the orbit $K \cdot x$ supports a K-invariant probability measure.

For any $x \in L/\Lambda$, put $K_x = \{\tau \in K : \tau(x) = x\}$. Let $\xi : K/K_x \to L/\Lambda$ be the map defined by $\xi(\tau K_x) = \tau(x)$ for all $\tau \in K$. Since $\mathrm{Aff}(L/\Lambda)$ acts continuously on L/Λ, we have that ξ is a continuous injective K-equivariant map. Let $x \in X_1$ be such that $K \cdot x$ supports a K-invariant

probability measure. Since ξ is injective, the measure can be lifted to a K-invariant probability measure on K/K_x. Let $\tau \in K_x$. Then for any $g \in G$, we have $\tau(gx) = g\tau(x) = gx$. Now since $\overline{Gx} = L/\Lambda$, we have that $\tau(y) = y$ for all $y \in L/\Lambda$. Hence K_x is the trivial subgroup of $\mathrm{Aff}(L/\Lambda)$. Thus K admits a finite Haar measure. Hence K is a compact group, and the claim is proved.

Let Ω be any neighbourhood of e in $Z_L(G)$. Put

$$R' = \{(x,y) \in R : y \notin K \cdot \Omega x\}.$$

Let X_c be the closure of the projection of R' on the first factor of $L/\Lambda \times L/\Lambda$. Put $X_0 = (L/\Lambda) \setminus X_c$.

Claim 6.6 $X_1 \subset X_0$.

Suppose the claim does not hold. Then there exists a sequence $\{(x_i, y_i)\} \subset R'$ converging to $(x,y) \in R$ with $x \in X_1$. By claim 6.4, there exists $\tau \in K$ such that $y = \tau(x)$. Therefore, after passing to a subsequence, there exists a sequence $g_i \to e$ in L such that $y_i = \tau(g_i x_i)$ for all $i \in \mathbb{N}$. By the definition of R', $g_i \notin \Omega \subset Z_L(G)$ for all $i \in \mathbb{N}$. Also $(x_i, g_i x_i) \in R$ for all $i \in \mathbb{N}$. By Ratner's theorem 4.2, there exits a $\Delta(G)$-invariant $\Delta(G)$-ergodic probability measure μ_i on $(L/\Lambda) \times (L/\Lambda)$ such that $\overline{\Delta(G)(x_i, g_i x_i)} = \mathrm{supp}(\mu_i)$. Let $h_i \to e$ be a sequence in L such that $x_i = h_i x$ for all $i \in \mathbb{N}$. By theorem 4.3, after passing to a subsequence, there exists a probability measure μ on $L/\Lambda \times L/\Lambda$ such that $\mu_i \to \mu$ as $i \to \infty$, and the following holds: $\mathrm{supp}(\mu) = F \cdot (x,x)$, where F is a closed subgroup of $L \times L$, and

$$({h_i}^{-1}, {h_i}^{-1}{g_i}^{-1})\Delta(G)(h_i, g_i h_i) \subset F, \qquad \forall i \in \mathbb{N}. \tag{16}$$

In particular, $F \cdot (x,x) \subset R$ and $\Delta(G) \subset F$. Since $x \in X_1$, we have that $F \supset \Delta(L)$. By an argument as in the proof of claim 6.4, we conclude that $F \cap \ker(p_i) = \{e\}$ for $i = 1, 2$. Therefore $F = \Delta(L)$. Hence from equation 16 we conclude that $g_i \in Z_L(G)$, which is a contradiction. This completes the proof of the claim, and the proof of the theorem. □

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Note added in the proof: Equidistributions of translates of measures is also considered in a recent preprint (later published as [EMM]) "Upper bounds and asymptotics in a quantitative version of the Oppenheim conjecture" of A Eskin, G A Margulis and S Mozes. In the context of the preprint it seems worthwhile to remark that the method in the present paper can be used to obtain 'uniform versions' of theorem 1.4 and corollary 1.2, as done in the above mentioned paper, for certain results.